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Building asymmetry into circular distributions

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ABSTRACT

Most of the tractable distributions currently available for modeling circular data are symmetric around a modal direction, prominent among them the von Mises distribution. Here we discuss a method of introducing asymmetry into any such symmetric circular model and develop general classes of non-symmetric circular distributions. In particular, we introduce and study a resulting variation of the classical von Mises distribution, along with a biological application.

1. Introduction

Directions in 2-dimensions can be represented as points on the circumference of a unit circle and models for representing such data are called circular distributions. See for instance Jammalamadaka and SenGupta (2001, Chapter 2) for a discussion of many such models. Most of these models are typically symmetric around some center and very few non-symmetric distributions are available for describing circular data. Some recent papers (see e.g. Jammalamadaka and Kozubowski (2003), Fernandez-Duran (2004), Gatto and Jammalamadaka (2007)) address this issue and provide classes of asymmetric circular models. In this paper, we describe a broad class of models for the non-symmetric case and discuss an application.

2. General method

As shown in Azzalini (2005), if f is density on the real line which is symmetric about a given point, say 0 without loss of generality, and G is a distribution function such that G' exists and is a density which is symmetric about 0, then it can be seen that $2f(x)G(w(x))$ is a density over the real line for any odd function w. The resulting density is typically non-symmetric. In Theorem 1, we adapt this idea to the circular case, resulting in asymmetric variations for any given symmetric circular model. Recall that a circular pdf is a non-negative periodic function with period 2π, which integrates to one over intervals of length 2π. For specificity, we will consider the distribution as being defined over the interval $[-\pi, \pi)$ in this discussion.

Theorem 1. *Suppose that f and g are circular densities which are symmetric about 0 and $G(\theta) = \int_{-\pi}^{\theta} g(\alpha)\mathrm{d}\alpha$. If w is an odd function with $|w(\theta)| \le \pi$ and periodic i.e. $w(\theta) = w(\theta + 2\pi k)$ for all integers k, then*

$$f_\mu(\theta) = 2f(\theta - \mu)G(w(\theta - \mu))$$

is a circular density.

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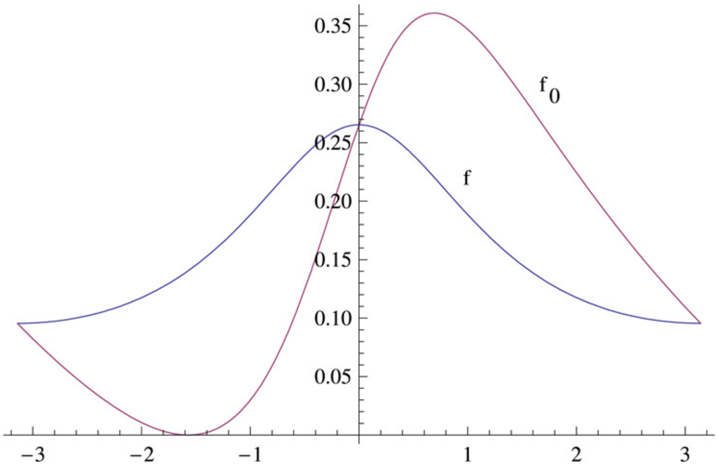

Fig. 1. Graphs of f, the wrapped Cauchy distribution with $\mu = 0$, and $\rho = 0.25$, and the generated distribution $f_0(\theta) = f(\theta)\,(1 + \sin\theta)$.

Proof. By Azzalini's result, f_0, (f_μ with $\mu = 0$) is a density on $[-\pi, \pi)$ because f is a density on $[-\pi, \pi)$ and $w(\theta) \in [-\pi, \pi)$ for $-\pi \le \theta < \pi$. But f_0 is also a circular density because

$$f_0(\theta + 2\pi k) = f(\theta + 2\pi k)G(w(\theta + 2\pi k)) = f(\theta)G(w(\theta)) = f_0(\theta)$$

for all integers k. The result is proved by noting that f_μ is a location change of f_0. □

The method can generally be used to obtain skewed modifications of circular distributions that are symmetric about some value, say μ, by observing that f_μ is the symmetric density $f(\theta - \mu)$ perturbed by the factor $2\,G(w(\theta - \mu))$. Different choices of G and w provide a wide selection of such densities. Note that if w is identically 0 then $f_\mu(\theta) = f(\theta - \mu)$ because the symmetry of g implies $G(0) = 1/2$.

An illuminating choice of G is the distribution function of the uniform distribution, $G(\theta) = (\pi + \theta)/2\pi$. For this case, we have

$$f_\mu(\theta) = f(\theta - \mu)\left(1 + \frac{w(\theta - \mu)}{\pi}\right).$$

Here we get some idea of how the original symmetric distribution is being perturbed by this method. Suppose, for example, that f is the density of the wrapped Cauchy distribution

$$f(\theta - \mu) = \frac{1}{2\pi}\,\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\theta - \mu)}$$

for $0 \le \rho < 1$ and $w(\theta) = \pi\ \sin\theta$. The density f_μ is given by

$$f_\mu(\theta) = \frac{1}{2\pi}\,\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\theta - \mu)}(1 + \sin(\theta - \mu)).$$

Fig. 1 contains a graph of the wrapped Cauchy distribution with $\rho = 0.25$ and $\mu = 0$ along with the generated distribution f_0. The method thus produces a distribution which exhibits a pronounced skew.

3. Parametric families

By incorporating one or more parameters in the definition of w, we can introduce families of distributions. Judicious choices of such parameters will lead to the original distribution as a member of the family with sufficient variation about it to produce a useful family for modeling purposes. For example, if $w(\theta) = \lambda\pi\sin(k\theta)$ for some integer k, then we get a family of distributions given by

$$2f(\theta - \mu)G(\lambda\pi\ \sin(k(\theta - \mu)))$$

for $-1 \le \lambda \le 1$. The choice of $\lambda = 0$ produces the original distribution because we must have $G(0) = 1/2$. Values of λ near zero will produce small perturbations of the original distribution, with the variation in shape increasing as $|\lambda|$ gets larger.

For example, again let $f(\theta - \mu)$ be the wrapped Cauchy distribution, G be the uniform distribution function, and $w(x) = \lambda\pi\ \sin(2\theta)$. This produces a family of distributions given by

$$f_\lambda(\theta;\rho,\ \mu,\ k) = \frac{1}{2\pi}\,\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\theta - \mu)}(1 + \lambda\sin(2(\theta - \mu))).$$

Fig. 2 contains graphs of the family members with $\mu = 0$, $\rho = 0.1$ and $\lambda = 0,\ 0.5,\ 1$.

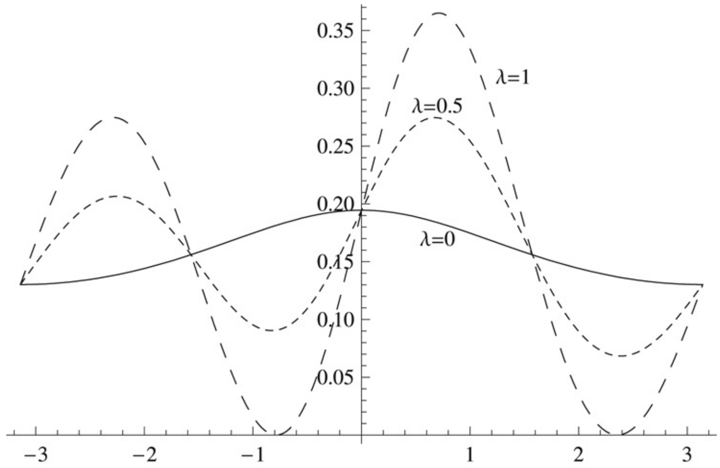

Fig. 2. Densities generated from the wrapped Cauchy distribution, $f_\lambda(\cdot\,; 0.1, 0, 2)$, for $\lambda = 0, 0.5$, and 1.

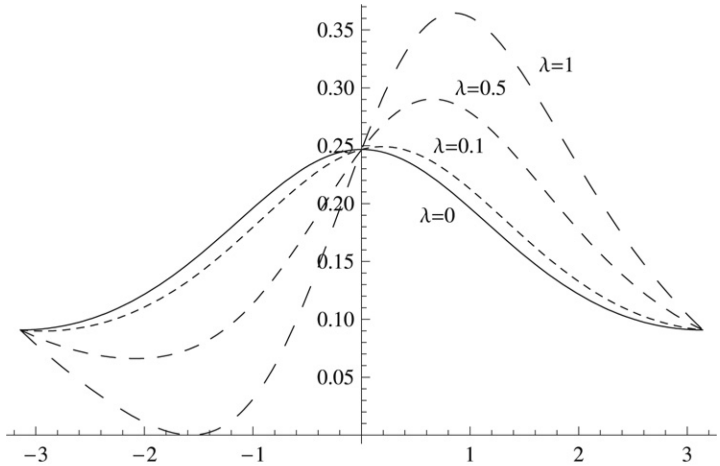

Fig. 3. Densities generated from the von Mises distribution, $v_\lambda(\cdot\,; 0, 0.5)$, for $\lambda = 0, 0.1, 0.5$, and 1.

4. An asymmetric version of the von Mises distribution

The von Mises, or circular normal, distribution plays a central role in the analysis of circular data. Its density is given by

$$v(\theta; \mu, \kappa) = \frac{e^{\kappa \cos(\theta-\mu)}}{2\pi I_0(\kappa)} \tag{1}$$

with parameters $0 \le \mu < 2\pi$ and $\kappa > 0$. Here I_0 is the modified Bessel function of the first kind of order 0. (Higher orders will be identified through the subscript.) This density is symmetric about the direction μ. The method presented in Section 3 will now be used to extend this family to include some skewed alternatives.

Again using the uniform distribution, we define a three parameter family of densities using $w(\theta) = \lambda\pi \sin\theta$ with $-1 \le \lambda \le 1$, given by

$$v_\lambda(\theta; \mu, \kappa) = \frac{e^{\kappa \cos(\theta-\mu)}}{2\pi I_0(\kappa)} \left(1 + \lambda \sin(\theta - \mu)\right). \tag{2}$$

The choice of $\lambda = 0$ yields the von Mises distribution. Other values of λ generate skewed alternatives to the von Mises distribution. Graphs of these distributions with $\mu = 0$, $\kappa = 0.5$, and $\lambda = 0, 0.1, 0.5$, and 1 are presented in Fig. 3. Negative values of λ skew the distribution in the opposite direction.

The trigonometric moments of these distributions are related to those of the von Mises distribution in an interesting way as described in Theorem 2.

Theorem 2. *With φ_p^* the pth trigonometric moment of the von Mises distribution in (1) and $\varphi_p(\lambda)$ the pth trigonometric moment of $\upsilon_\lambda(\cdot\,; \mu, \kappa, \lambda)$ in (2), we have*

$$\varphi_p(\lambda) = \varphi_p^* + \mathrm{i}(\lambda/2)\mathrm{e}^{\mathrm{i}p\mu}\left(\varphi_{p-1}^* - \varphi_{p+1}^*\right) \quad \textit{for } p = 0, \pm 1, \pm 2, \ldots.$$

Proof.

$$\begin{aligned}
\varphi_p(\lambda) &= \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta} f(\theta; \mu, \kappa, \lambda)\mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta} \frac{\mathrm{e}^{\kappa\cos(\theta-\mu)}}{2\pi I_0(\kappa)} (1 + \lambda \sin(\theta - \mu))\, \mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta} \frac{\mathrm{e}^{\kappa\cos(\theta-\mu)}}{2\pi I_0(\kappa)} \mathrm{d}\theta + \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta} \lambda \sin(\theta - \mu) \frac{\mathrm{e}^{\kappa\cos(\theta-\mu)}}{2\pi I_0(\kappa)} \mathrm{d}\theta \\
&= \varphi_p^* + \mathrm{e}^{\mathrm{i}p\mu} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta} \lambda \sin(\theta) \frac{\mathrm{e}^{\kappa\cos(\theta)}}{2\pi I_0(\kappa)} \mathrm{d}\theta.
\end{aligned}$$

Now, using $\sin(\theta) = \mathrm{i}(\mathrm{e}^{-\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta})/2$, we can rewrite the above as

$$\begin{aligned}
\varphi_p(\lambda) &= \varphi_p^* + \mathrm{i}(\lambda/2)\mathrm{e}^{\mathrm{i}p\mu} \int_0^{2\pi} \left(\mathrm{e}^{\mathrm{i}(p-1)\theta} - \mathrm{e}^{\mathrm{i}(p+1)\theta}\right) \frac{\mathrm{e}^{\kappa\cos(\theta)}}{2\pi I_0(\kappa)} \mathrm{d}\theta \\
&= \varphi_p^* + \mathrm{i}(\lambda/2)\mathrm{e}^{\mathrm{i}p\mu}\left(\varphi_{p-1}^* - \varphi_{p+1}^*\right). \quad \square
\end{aligned}$$

Using the results from page 38 of Jammalamadaka and SenGupta (2001), we may write

$$\varphi_p^* = A_p(\kappa)\mathrm{e}^{\mathrm{i}p\mu},$$

where $A_p(\kappa) = I_p(\kappa)/I_0(\kappa)$. Thus, we find that the first trigonometric moment is given by

$$\varphi_1(\lambda) = A_1(\kappa)\mathrm{e}^{\mathrm{i}\mu} + \mathrm{i}(\lambda/2)\mathrm{e}^{\mathrm{i}\mu}\left(1 - A_2(\kappa)\mathrm{e}^{\mathrm{i}2\mu}\right).$$

Since we have a location model, the action on the first trigonometric moment from transforming the distribution can be described though its action on any particular value of μ, say $\mu = 0$, for which

$$\varphi_1(\lambda) = A_1(\kappa) + \mathrm{i}(\lambda/2)\left(1 - A_2(\kappa)\right).$$

The length of $\varphi_1(\lambda)$ is

$$\sqrt{A_1(\kappa)^2 + \frac{\lambda^2}{4}\left(1 - A_2(\kappa)\right)^2}.$$

Thus, we see that the length of this vector is greater than the length of φ_1^*, which is $A_1(\kappa)$. This is consistent with the linear skew-symmetric results in that the skewed distribution has less variability than that of the original distribution. The angle associated with $\varphi_1(\lambda)$ is given by

$$\alpha(\lambda) = \arctan\left(\frac{(\lambda/2)(1 - A_2(\kappa))}{A_1(\kappa)}\right).$$

Thus the mean direction of $f(\cdot\,; \mu, \kappa, \lambda)$ is μ rotated by $\alpha(\lambda)$. Now, $A_1(\kappa)$ and $(1 - A_2(\kappa))$ are both positive for $\kappa > 0$. In addition, it can be shown that $(1 - A_2(\kappa))/A_1(\kappa)$ is a decreasing function of κ with

$$\lim_{\kappa \to 0^+} \frac{1 - A_2(\kappa)}{A_1(\kappa)} = \frac{\pi}{2}.$$

Thus, the rotation about μ, namely $\alpha(\lambda)$, will be counterclockwise for $0 < \lambda \le 1$ and clockwise for $-1 \le \lambda < 0$ with the amount of rotation increasing with $|\lambda|$. This is consistent with linear skew-symmetric distributions in that the mean is an increasing function of the skewing parameter. See also Umbach (2006).

5. An application

Rudolf Jander's experiments concerning the direction chosen by ants in response to a stimulus has long provided some interesting problems in modeling of circular distributions. See Jander (1957) for the original description of these experiments. In particular, the data set of Example 4.4 on page 60 of Fisher (1993) has generated much interest. Fisher clearly demonstrates that the von Mises distribution does not fit this data well. We come closer to an accurate model using the asymmetric version of the von Mises distribution presented Section 4.

The first step is to find the maximum likelihood estimates of the parameters μ, κ, and λ in model (2). This was carried out with *Mathematica* with the following results in radian measure over $(0, 2\pi)$:

$$\hat{\mu} = 2.88081 \tag{3}$$

$$\hat{\kappa} = 1.4361 \tag{4}$$

$$\hat{\lambda} = 0.523646. \tag{5}$$

Again using *Mathematica*, we find the maximum likelihood estimates of the parameters μ and κ of the von Mises distribution, model (1), as:

$$\hat{\mu} = 3.19637 \tag{6}$$

$$\hat{\kappa} = 1.55763. \tag{7}$$

The last values compare well with the results in Fisher (1993) that $\hat{\mu} = 183.1^\circ = 3.19570$ and $\hat{\kappa} = 1.54$.

To check the goodness of fit of models (1) and (2), we computed both the Kuiper and Watson statistics for each model. These tests may be found in Jammalamadaka and SenGupta (2001). We computed these under the assumption that the estimated values in (3)–(7) were the exact values, this assumption being justified by rather large sample size of $n = 100$. Using the values in (3)–(5), the Kuiper statistic for model (2) has the value $V_{100} = 11.2918$. Using the values in (6) and (7), the Kuiper statistic for model (1) has the value $V_{100} = 12.5958$. The smaller value of the statistics for model (2) indicates a better fit. Using the values in (3)–(5), the Watson statistic for model (2) has the value $W^2_{100} = 0.296594$. Using the values in (6) and (7), the Watson statistic for model (1) has the value $W^2_{100} = 0.326112$. The smaller value of the statistics for model (2) indicates a better fit. Thus, we see that by either measure, the asymmetric model (2) provides a better fit to the data than the von Mises distribution (1).

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